

O-MINIMAL STRUCTURES: LOW ARITY VERSUS GENERATION

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ABSTRACT. We show that an analogue of the Hilbert's Thirteenth Problem fails in the real subanalytic setting. Namely we prove that, for any integer n , the o -minimal structure generated by restricted analytic functions in n variables is strictly smaller than the structure of all global subanalytic sets, whereas these two structures define the same subsets in \mathbb{R}^{n+1} .

1. INTRODUCTION

The aim of this paper is to prove that, for any fixed $n \in \mathbb{N}$, the o -minimal structure generated by the family of all global subanalytic subsets of \mathbb{R}^n is strictly smaller than the structure of all global subanalytic sets: some subanalytic subsets of \mathbb{R}^{n+1} are “transcendental” over the family of all subanalytic subsets of \mathbb{R}^n .

The main motivation for this work was to prove that the statement

“Given an o -minimal structure \mathcal{S} over X , there is an integer n such that \mathcal{S} and $\text{str}(\mathcal{S}^{(n)})$ - its reduct generated by \mathcal{S} -definable subsets of X^n - define the same subsets of X^N , for all N ”

is false. We now know it fails for \mathcal{S} being the structure of global subanalytic sets.

This result can be seen as a negative answer to a generalized real analytic version of the second part of Hilbert's Thirteenth Problem: subanalytic functions don't have the superposition property (see [12] for the positive answer in the continuous setting).

In section 2, we give the following definitions: o -minimal structure, generated structure, subanalytic sets and sub- n -analytic sets; only the last one is original. We then recall some well known properties.

In section 3 is proven that restricted analytic functions in n variables and subanalytic subsets of \mathbb{R}^{n+1} have the same definability power. This elegant proof is due to Daniel J. Miler and is based on Hironaka's Uniformization Theorem for subanalytic sets.

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In section 4-7, we use Gabrielov's "Explicit Fibre Cutting Lemma", a diagonal argument on formal series and metric control on truncation of translated power series, to prove that there is a restricted analytic function $f : [-1, 1]^{n+1} \rightarrow \mathbb{R}$ whose graph can't be defined by mean of restricted analytic functions in n variables.

2. DEFINITIONS

Definition 2.1. We call $\mathcal{S} = (\mathcal{S}^{(n)})_{n \in \mathbb{N}}$ a structure over $(\mathbb{R}; +, \cdot)$ if it has the following properties

- (S1) $\mathcal{S}^{(n)}$ is a boolean subalgebra of $\mathcal{P}(\mathbb{R}^n)$ for each $n \in \mathbb{N}$,
- (S2) if n is an integer and A is a semialgebraic subset of \mathbb{R}^n then $A \in \mathcal{S}^{(n)}$,
- (S3) if $A \in \mathcal{S}^{(n)}$, then $\mathbb{R} \times A \in \mathcal{S}^{(n+1)}$
- (S4) if $A \in \mathcal{S}^{(n+1)}$ and $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the cartesian projection $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$ then $\pi(A) \in \mathcal{S}^{(n)}$

If it furthermore has the property:

- (S5) every element of $\mathcal{S}^{(1)}$ is a finite union of singletons and open intervals,

it is said to be an o -minimal structure over $(\mathbb{R}; +, \cdot)$.

In words, a structure over $(\mathbb{R}; +, \cdot)$ is a collection of real sets, containing the family of all semialgebraic sets and stable under natural set theoretical operations: union, intersection, complementation, cartesian projection and cartesian product. The structure is o -minimal if the elements of $\mathcal{S}^{(1)}$ are the simplest possible: finite union of intervals and points.

Elements of $\bigcup_n \mathcal{S}^{(n)}$ are called \mathcal{S} -definable sets; given a \mathcal{S} -definable set A , we call the integer n such that $A \in \mathcal{S}^{(n)}$ the *arity* of A .

A function f from some $A \subseteq \mathbb{R}^n$ to \mathbb{R}^m is said to be \mathcal{S} -definable if its graph is a \mathcal{S} -definable set.

For an introduction to the geometry in o -minimal structure, see for instance [6] or [7].

Let us now define the notion of *generated structure*.

If $\mathcal{U} = (\mathcal{U}^{(n)})_{n \in \mathbb{N}}$ and $\mathcal{V} = (\mathcal{V}^{(n)})_{n \in \mathbb{N}}$ are such that $\mathcal{U}^{(n)} \subseteq \mathcal{P}(\mathbb{R}^n)$ and $\mathcal{V}^{(n)} \subseteq \mathcal{P}(\mathbb{R}^n)$, we will note $\mathcal{U} \sqsubseteq \mathcal{V}$ the property " $\mathcal{U}^{(n)} \subseteq \mathcal{V}^{(n)}$ for all $n \in \mathbb{N}$ ".

If $\mathcal{A} = (\mathcal{A}^{(n)})_{n \in \mathbb{N}}$ is such that $\mathcal{A}^{(n)} \subseteq \mathcal{P}(\mathbb{R}^n)$, there exists a smallest element - for the partial order \sqsubseteq on $\prod_{n \in \mathbb{N}} \mathcal{P}(\mathcal{P}(\mathbb{R}^n))$ - among the $\mathcal{S} = (\mathcal{S}^{(n)})_{n \in \mathbb{N}}$ forming a structure over $(\mathbb{R}; +, \cdot)$ and satisfying $\mathcal{A} \sqsubseteq \mathcal{S}$. We will note this structure $\text{str}(\mathcal{A})$, and call it the *structure generated by* \mathcal{A} .

Remark 2.2. Let n_0 be an integer and $\mathcal{F}^{(n_0)}$ a subset of $\mathcal{P}(\mathbb{R}^{n_0})$; we will, when no confusion is possible, identify $\mathcal{F}^{(n_0)}$ and the family

$$\mathcal{G} = (\mathcal{G}^{(n)})_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathcal{P}(\mathcal{P}(\mathbb{R}^n)),$$

where $\mathcal{G}^{(n)} = \emptyset$ if $n \neq n_0$ and $\mathcal{G}^{(n_0)} = \mathcal{F}^{(n_0)}$.

In such a case $\text{str}(\mathcal{F}^{(n_0)})$ stands for $\text{str}(\mathcal{G})$.

Given an $n \in \mathbb{N}$, we let $\mathcal{B}(n)$ be the algebra of all functions $f : [-1, 1]^n \rightarrow \mathbb{R}$ such that f admits an analytical continuation in a neighbourhood of $[-1, 1]^n$. We call such a function f a *restricted analytic function* (in n variables).

Let $\mathcal{E} = (\mathcal{E}^{(n)})_{n \in \mathbb{N}^*}$ be the element of $\prod_{n \in \mathbb{N}^*} \mathcal{P}(\mathcal{P}(\mathbb{R}^n))$ defined by

$$\mathcal{E}^{(n+1)} := \{\text{graph}(f), f \in \mathcal{B}(n)\}.$$

With the previous notation, we denote by \mathbb{R}_{an} the structure $\text{str}(\mathcal{E})$.

Theorem 2.3 (Gabrielov). \mathbb{R}_{an} is an *o-minimal structure*.

An element A in \mathbb{R}_{an} is called a *global subanalytic set*.

Definition 2.4. Given an integer n we let

$$\mathbb{R}_{\text{an}(n)} := \text{str}(\mathcal{E}^{(n+1)});$$

$\mathbb{R}_{\text{an}(n)}$ -definable sets are called *global sub- n -analytic sets*.

In words, $\mathbb{R}_{\text{an}(n)}$ is the structure generated by the graphs of all restricted analytic functions *in at most n variables* (whereas there is no bound on the number of variables for the restricted analytic functions used to generate \mathbb{R}_{an}).

For instance,

$$\{(x_1, x_2, x_3) \in [-1, 1]^3; \cos \frac{x_1 + x_2}{2} + \sin \frac{x_3 - \cos x_2}{2} > 0\}$$

is a $\mathbb{R}_{\text{an}(1)}$ -definable subset of \mathbb{R}^3 .

Proposition 2.5. $\mathbb{R}_{\text{an}(n)}$ is model complete (as a $\mathcal{B}(n)$ -structure).

Let p be an integer; we will denote by $A_p(\mathcal{B}(n))$ the subalgebra of $\mathcal{B}(p)$ generated by all the functions

$$(x_1, \dots, x_p) \mapsto f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

as σ ranges in $\{1, \dots, p\}^{\{1, \dots, n\}}$ (the set of functions from $\{1, \dots, n\}$ to $\{1, \dots, p\}$) and f ranges in $\mathcal{B}(n)$ (the set of restricted analytic functions in n variables).

Once we have noted that, for every $p \in \mathbb{N}$, the algebra $A_p(\mathcal{B}(n))$ is stable under the action of partial derivation operators, Proposition 2.5 easily follows from Gabrielov's "Explicit Model Completeness" ([11], Theorem 1. and Corollary).

We will use a more precise version of this result in sections 4 and 5 to show how $\mathbb{R}_{\text{an}(n)}$ -definable functions are controlled by restricted analytic functions in at most n variables.

3. SUB-N-ANALYTIC SETS

Proposition 3.1. *$\mathbb{R}_{\text{an}(n)}$ is the structure generated by global subanalytic sets of arity $n + 1$.*

(Proof due to Daniel J. Miller.)

The inclusion $\mathbb{R}_{\text{an}(n)} \subseteq \text{str}(\mathbb{R}_{\text{an}}^{(n+1)})$ is easy.

Let us prove the other inclusion by induction on n ; the case $n = 0$ is clear. Let denote by K the set $[-1, 1]^n$. By the cell decomposition theorem ([7], theorem 2.11.), it's enough to prove that, given a \mathbb{R}_{an} -definable function $f : C \rightarrow \mathbb{R}$ for C being a \mathbb{R}_{an} -cell either included in or disjoint with K , then f is $\mathbb{R}_{\text{an}(n)}$ -definable.

Note that the mapping $i : (x_1, \dots, x_n) \mapsto (1/x_1, \dots, 1/x_n)$ is $\mathbb{R}_{\text{an}(n)}$ -definable and sends $\mathbb{R} \setminus K$ in K ; we thus can suppose that $A \subseteq K$.

Up to a finer cell decomposition, we can furthermore suppose that $|f(\bar{x})| - 1$ has constant sign on C and, $y \mapsto 1/y$ being $\mathbb{R}_{\text{an}(n)}$ -definable, we can assume that $|f(\bar{x})| \leq 1$ for all $\bar{x} \in C$.

Let G be the closure of the graph of f ; G is a compact subanalytic set of dimension $d \leq n$.

Hironaka's uniformization theorem ([1], theorem 0.1.) gives a d -dimensional analytic manifold Y and a surjective analytic proper mapping $\psi : Y \rightarrow G$.

G being compact and ψ being surjective and proper, Y is compact; we then easily get a finite family $\{\phi_i : [-1, 1]^d \rightarrow Y\}_{i=1, \dots, s}$ of restricted analytic functions such that the union of their images is covering Y .

Hence $G = \bigcup_{i=1}^s \psi \circ \phi_i([-1, 1]^d)$ is a $\mathbb{R}_{\text{an}(d)}$ -definable set and thus a $\mathbb{R}_{\text{an}(n)}$ -definable set.

By induction hypothesis, C is an $\mathbb{R}_{\text{an}(n-1)}$ -definable set and thus a $\mathbb{R}_{\text{an}(n)}$ -definable set. The function f is $\mathbb{R}_{\text{an}(n)}$ -definable, for its graph, $G \cap (C \times \mathbb{R})$, is.

4. N-REGULARITY

In the following sections, we prove that there are some \mathbb{R}_{an} -definable analytic functions in $n + 1$ variables which are not $\mathbb{R}_{\text{an}(n)}$ -definable. We first show how each $\mathbb{R}_{\text{an}(n)}$ -definable function is “controlled”, through the notion of *n-regularity*, by the restricted analytic functions in n variables used to define it.

Let n and p be two integers; in the proof of the Proposition 2.5, we have defined the algebra $A_p(\mathcal{B}(n))$.

By definition, each $g \in A_p(\mathcal{B}(n))$ can be written in the form

$$g(x_1, \dots, x_p) = Q(h_1(x_{\sigma_1(1)}, \dots, x_{\sigma_1(n)}), \dots, h_q(x_{\sigma_q(1)}, \dots, x_{\sigma_q(n)}))$$

where q is an integer, Q is a polynomial in q variables with integer coefficients, the h_i 's are restricted analytic functions in n variables and the σ_i 's are mappings from $\{1, \dots, n\}$ to $\{1, \dots, p\}$.

We will call an element of $A_p(\mathcal{B}(n))$ a *restricted analytic function in p variables which essentially depends on at most n variables*.

In some sense, the graph of a $\mathbb{R}_{\text{an}(n)}$ -definable function looks almost everywhere like an analytic variety defined as a zero-set of restricted analytic functions depending on at most n variables.

Let's make this statement more precise: we first recall a special case of Gabrielov's "Explicit Fibre Cutting Lemma" (see [11], Lemma 3. and Theorem 1.):

Theorem 4.1 (Gabrielov). *Given a d -dimensional sub- n -analytic set $Y \subseteq \mathbb{R}^m$, there is a $p \in \mathbb{N}$, a finite family $\{X_\nu\}$ of sub- n -analytic subsets of \mathbb{R}^{m+p} and a sub- n -analytic subset V of \mathbb{R}^{m+p} such that, if $\pi : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ is given by $\pi(x_1, \dots, x_{m+p}) = (x_1, \dots, x_m)$, one has*

- (1) $Y = \pi(V) \cup \bigcup \pi(X_\nu)$;
- (2) $\dim \pi(V) < d$;
- (3) for each ν , $\dim X_\nu = d$ and $\pi|_{X_\nu} : X_\nu \rightarrow Y$ has rank d at every point of X_ν ;
- (4) for each $\bar{s} \in X_\nu$, $\{\bar{x} - \bar{s}; \bar{x} \in X_\nu\}$ is near $\bar{0}$ the zero-set of $m+p-d$ elements $f_i : \mathbb{R}^{m+p} \rightarrow \mathbb{R}$ of $A_{m+p}(\mathcal{B}(n))$, $(df_i)_i$ having rank $m+p-d$ at $\bar{0}$;
- (5) $X_\lambda \cap X_\mu = \emptyset$ for $\lambda \neq \mu$.

This theorem leads us to the following definition:

Definition 4.2. *Let f be a function, from a neighbourhood U of $\bar{0}$ in \mathbb{R}^{n+1} , to \mathbb{R} .*

f is said to be n -regular at $\bar{0}$ if there exist

- *an integer p ,*
- *a $(p+1)$ -tuple (g_1, \dots, g_{p+1}) of elements of $A_{n+p+2}(\mathcal{B}_n)$,*
- *a neighbourhood $V \subseteq U$ of $\bar{0} \in \mathbb{R}^{n+1}$ and*
- *for each $\bar{x} \in V$, there is a point $(y_1(\bar{x}), \dots, y_p(\bar{x}))$ in \mathbb{R}^p ,*

such that

- *$g_i(\bar{x}, y_1(\bar{x}), \dots, y_p(\bar{x}), f(\bar{x})) = 0$, for all i , and*
- *the rank of*

$$\left(\frac{\partial g_i}{\partial z_j} \right)_{\substack{1 \leq i \leq p+1 \\ n+2 \leq j \leq n+p+2}}$$

is full at the point $(\bar{x}, y_1(\bar{x}), \dots, y_p(\bar{x}), f(\bar{x}))$.

A function f from a neighbourhood U of $\bar{a} \in \mathbb{R}^{n+1}$ is said to be n -subregular at \bar{a} if $\bar{x} \mapsto f(\bar{a} + \bar{x})$ is n -regular at $\bar{0}$.

In words, f is n -regular at $\bar{0}$ if, as in Theorem 4.1, the germ of its graph is the germ of the projection $\pi(X)$ of an analytic manifold X given as the zero-set of some functions depending essentially on at most n variables, and $\pi|_X$ is locally a diffeomorphism.

Proposition 4.3. *Given a $\mathbb{R}_{\text{an}(n)}$ -definable function $f : [-1, 1]^{n+1} \rightarrow \mathbb{R}$, there is a point $\bar{a} \in]-1, 1[^n$ such that f is n -regular at \bar{a} .*

This proposition follows from an easy dimensional argument and Theorem 4.1.

5. DIAGONALIZATION

In the sequel we will build a function $h : [-1, 1]^{n+1} \rightarrow \mathbb{R}$ such that

- there is no $\bar{a} \in]-1, 1[^{n+1}$ at which h is n -regular (and thus h can't be $\mathbb{R}_{\text{an}(n)}$ -definable)
- but h is a restriction to $[-1, 1]^{n+1}$ of some analytic function from \mathbb{R}^{n+1} to \mathbb{R} (and subsequently is \mathbb{R}_{an} -definable).

We will now “enumerate” the germs (above $\bar{0} \in \mathbb{R}^{n+1}$) of n -regular (at $\bar{0}$) functions $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$.

We first have to choose a value y for $f(0, \dots, 0)$.

By definition of n -regularity, it is enough to look, as p ranges in \mathbb{N} , at all the $(p+1)$ -tuples (g_1, \dots, g_{p+1}) of elements in $\mathcal{A}_{n+p+2}(\mathcal{B}_n)$ such that $g_i(0, \dots, 0, y) = 0$ and the rank of

$$\left(\frac{\partial g_i}{\partial z_j} \right)_{\substack{1 \leq i \leq p+1 \\ n+2 \leq j \leq n+p+2}}$$

is full at points $(0, \dots, 0, y)$.

Let's fix such a $p \in \mathbb{N}$.

But by definition, each $g \in \mathcal{A}_{n+p+2}(\mathcal{B}_n)$ is of the following form:

- there is a $q \in \mathbb{N}$ and a $Q \in \mathbb{Z}[T_1, \dots, T_q]$,
- there are some h_1, \dots, h_q in $\mathcal{B}(n)$,
- for each $i \in \{1, \dots, q\}$, there is a mapping σ_i from $\{1, \dots, n\}$ to $\{1, \dots, n+p+2\}$

such that

$$g(x_1, \dots, x_{n+p+2}) = Q(h_1(x_{\sigma_1(1)}, \dots, x_{\sigma_1(n)}), \dots, h_q(x_{\sigma_q(1)}, \dots, x_{\sigma_q(n)})).$$

So let's fix a $q \in \mathbb{N}$ a $(p+1)$ -tuple of elements in $\mathbb{Z}[T_1, \dots, T_q]$ and for each $1 \leq j \leq q$ and $1 \leq i \leq p+1$, fix a mapping σ_j^i from $\{1, \dots, n\}$ to $\{1, \dots, n+p+2\}$.

The only parameters left free are now

- the value of y of $f(0, \dots, 0)$,
- the $(p+1)q$ -tuple of restricted analytic functions h in n variables.

All those germs are thus built by choosing a set of “assembly instructions” (the integers p and q , polynomials Q and mappings σ) and then by assembling “pieces” (the restricted analytic functions h in n variables) that fit this set of instructions.

Let

$$s \mapsto \left((p(s), q(s)), (Q_k(s))_{1 \leq k \leq p(s)+1}, (\sigma_j^i(s))_{\substack{1 \leq j \leq q(s) \\ 1 \leq i \leq p(s)+1}} \right)$$

be a surjective mapping from \mathbb{N} to

$$\coprod_{(p,q) \in \mathbb{N}^2} \{(p, q)\} \times (\mathbb{Z}[T_1, \dots, T_q])^{p+1} \times ((\{1, \dots, n+p+2\}^{\{1, \dots, n\}})^q)^{p+1}.$$

Fix an $s \in \mathbb{N}$ (and thus some integers $p(s)$, $q(s)$, some polynomials $(Q_k(s))_{1 \leq k \leq p(s)+1}$ and some mappings $(\sigma_j^k(s))_{\substack{1 \leq j \leq q(s) \\ 1 \leq k \leq p(s)+1}}$).

Then let M_s be the subset of

$$\mathbb{R} \times (\mathbb{R}\{X_1, \dots, X_n\}^{q(s)})^{p(s)+1}$$

which elements

$$\left(y, ((g_j^k)_{1 \leq j \leq q(s)})_{1 \leq k \leq p(s)+1} \right)$$

satisfy the conditions in Definition 4.2:

- (1) $h_i(0, \dots, 0, y) = 0, \forall i \in \{1, \dots, p(s)+1\}$
- (2) the rank of $(\frac{\partial h_i}{\partial x_j})_{\substack{1 \leq i \leq p+1 \\ n+1 \leq j \leq n+p+2}}$ at $(0, \dots, 0, y)$ is full,

with

$$h_k(x_1, \dots, x_{n+p(s)+2}) = Q_k(s)(g_1^k(\bar{x}^{\sigma(s)_1^k}), \dots, g_{q(s)}^k(\bar{x}^{\sigma(s)_{q(s)}^k})),$$

and

$$\bar{x}^{\sigma(s)_j^k} = (x_{\sigma(s)_j^k(1)}, \dots, x_{\sigma(s)_j^k(n)}).$$

Then by Implicit Function Theorem, we have a mapping

$$\Phi^s : M_s \longrightarrow \mathbb{R}\{Y_1, \dots, Y_{n+1}\}$$

which, sends

$$\left(y, (g_j^k)_{\substack{1 \leq j \leq q(s) \\ 1 \leq k \leq p(s)+1}} \right)$$

to the analytic function f defined in a neighbourhood of $\bar{0} \in \mathbb{R}^{n+1}$, satisfying

- $f(0, \dots, 0) = y$
- there are analytic functions $(f_1, \dots, f_{p(s)})$ in a neighbourhood of $(0, \dots, 0)$ such that the graph of $(f_1, \dots, f_{p(s)}, f)$ is, in a neighbourhood of $(0, \dots, 0, y)$, the zero-set of the h_i 's.

Remark 5.1. By the definition of n -regularity, if $f : U \rightarrow \mathbb{R}$ is n -regular at $\bar{0} \in \mathbb{R}^{n+1}$ then the germ of f at $\bar{0}$ is in $\bigcup_{s \in \mathbb{N}} \Phi^s(M_s)$.

Let's denote by $\mathbb{R}_{D,E}[X_1, \dots, X_m]$ the set of polynomials in k variables, of degree $< D$ and of order $\geq d$ at the origin, with real coefficients.

Definition 5.2. We denote the truncation mapping by

$$T_{DE}^m : \mathbb{R}\{X_1, \dots, X_m\} \rightarrow \mathbb{R}_{D,E}[X_1, \dots, X_m]$$

$$h \mapsto \sum_{D \leq |\nu| < E} \frac{\partial^{|\nu|} h}{\partial \bar{X}^\nu}(\bar{0}) \cdot \bar{X}^\nu.$$

The chain derivation rule and an easy induction on E gives us the next proposition, which will be useful to deduce non-surjectivity of the Φ^s from the non-surjectivity of some rational mapping Φ_{DE}^s between finite dimensional spaces.

Proposition 5.3. Given three integers s , D and E with $D < E$, let \widetilde{M}_s be the image of M_s by the truncation $\Pi := \text{Id} \otimes (T_{0E}^n \otimes q(s))^{\otimes (p(s)+1)}$ of power series:

$$\Pi : \mathbb{R} \times (\mathbb{R}\{X_1, \dots, X_n\}^{q(s)})^{p(s)+1} \rightarrow \mathbb{R} \times (\mathbb{R}_{0,E}[X_1, \dots, X_n]^{q(s)})^{p(s)+1}.$$

Then there is a rational mapping Φ_{DE}^s such that the following diagram

$$\begin{array}{ccc} M_s & \xrightarrow{\Phi^s} & \mathbb{R}\{Y_1, \dots, Y_{n+1}\} \\ \Pi \downarrow & & \downarrow T_{DE}^{n+1} \\ \widetilde{M}_s & \xrightarrow{\Phi_{DE}^s} & \mathbb{R}_{D,E}[Y_1, \dots, Y_{n+1}] \end{array}$$

is commutative.

This proposition simply says that the derivatives at the origin of order $< E$ of an element ξ in the image of Φ^s depend only on y and on the derivatives at the origin of order $< E$ of g_j^k used to define ξ in the source space of Φ^s , and this in a rational manner.

6. TRANSLATION IN THE SOURCE SPACE

The previous section would help us produce, by a diagonal argument, an analytic function which is outside of the image of each Φ^s and thus is not n -regular at $\bar{0} \in \mathbb{R}^{n+1}$.

But what we want is a function which is *nowhere* n -regular in a neighbourhood of $\bar{0}$. Hence we have to look at $\bar{x} \mapsto h(\bar{\alpha} + \bar{x})$ as $\bar{\alpha}$ ranges in a neighbourhood (let's say $] -1, 1[^{n+1}$) of $\bar{0}$; unfortunately, we lose the finite dimensional dependency we found in the previous section.

More precisely, for $\bar{\alpha} \in] -1, 1[^{n+1}$, if we let $\tau_{\bar{\alpha}}$ be the function that assigns to an analytic function h near $[-1, 1]^{n+1}$ the function $\bar{x} \mapsto h(\bar{x} + \bar{\alpha})$ (which is analytic near $\bar{0}$), we don't have the equality

$$T_{DE}^{n+1}(\tau_{\bar{\alpha}}(h)) = T_{DE}^{n+1}(\tau_{\bar{\alpha}}(T_{DE}^{n+1}(h)));$$

each partial derivative of $h_{\bar{\alpha}}$ at the origin depends on *all* partial derivatives of h at zero.

The aim of this section is to show that this dependency can however be handled by metric arguments.

We first equip each $\mathbb{R}_{D,E}[Y_1, \dots, Y_{n+1}]$ with the norm:

$$\left\| \sum_{\nu} a_{\nu} \bar{Y}^{\nu} \right\|_{\infty} = \max_{\nu} \{|a_{\nu}|\}.$$

Proposition 6.1. *Let (D_k) be a increasing sequence of integers, η a positive real number, $\bar{\alpha}$ a point in $] -1, 1[^{n+1}$, h an analytic function in a neighbourhood of $[-1, 1]^{n+1}$ and K an integer.*

If for all $k > K$, we have

$$\|T_{D_k D_{k+1}}^{n+1}(h)\|_{\infty} \leq \frac{\eta}{2^k (D_{k+1}!)^{n+1}},$$

then

$$\|T_{D_K D_{K+1}}^{n+1}(\tau_{\bar{\alpha}}(h)) - T_{D_K D_{K+1}}^{n+1}(\tau_{\bar{\alpha}}(T_{D_K D_{K+1}}^{n+1}(h)))\|_{\infty} \leq \eta.$$

This is an easy consequence of the fact that, if $D_k \leq |\mu| < D_{k+1}$, then

$$\frac{\partial^{|\mu|}(\tau_{\bar{\alpha}}(h))}{\partial Y_1^{\mu_1} \dots \partial Y_{n+1}^{\mu_{n+1}}}(\bar{0}) = \sum_{j \geq k} \sum_{\substack{\nu_i \geq \mu_i \\ D_j \leq |\nu| < D_{j+1}}} \frac{\partial^{|\nu|} h}{\partial Y_1^{\nu_1} \dots \partial Y_{n+1}^{\nu_{n+1}}}(\bar{0}) \cdot \prod_i \binom{\nu_i}{\mu_i} \alpha_i^{\nu_i - \mu_i}$$

and $|\prod_i \binom{\nu_i}{\mu_i} \alpha_i^{\nu_i - \mu_i}| \leq (D_{k+1}!)^{n+1}$ if $|\nu| < D_{k+1}$ and $|\bar{\alpha}| \leq 1$.

Remark 6.2. *The linear mapping $L_{\bar{\alpha}}^k$ on $\mathbb{R}_{D_k, D_{k+1}}[Y_1, \dots, Y_{n+1}]$ defined by $L_{\bar{\alpha}}^k(P) = T_{D_k D_{k+1}}(\tau_{\bar{\alpha}}(P))$ is an isomorphism, since the image of a monomial \bar{X}^{ν} is the sum of \bar{X}^{ν} and some lower degree monomials.*

Furthermore we have the identity

$$\|(L_{\bar{\alpha}}^k)^{-1}\|_{\infty} = \max\{1/\|L_{\bar{\alpha}}^k(P)\|_{\infty}; \|P\|_{\infty} = 1\}$$

and the mapping $(P, \bar{\alpha}) \mapsto 1/\|L_{\bar{\alpha}}^k(P)\|_{\infty}$ is continuous on the compact set $\{\|P\|_{\infty} = 1\} \times [-1, 1]^{n+1}$.

Thus we have a bound S_k for the norm of $(L_{\bar{\alpha}}^k)^{-1}$, independent on $\bar{\alpha} \in] -1, 1[^{n+1}$.

7. CONSTRUCTION

We will use the good behaviour through truncation of the Φ^s to build a sequence of integers (D_s) and, for each $s \in \mathbb{N}$, a polynomial P_s in $\mathbb{R}_{D_s, D_{s+1}}[Y_1, \dots, Y_n]$, such that the formal power $h(Y_1, \dots, Y_{n+1}) = \sum_s P_s(Y_1, \dots, Y_{n+1})$ is the power expansion of an analytic function on \mathbb{R}^{n+1} but such that $\tau_{\bar{\alpha}}(h)$ is outside of the image of Φ^s for each $s \in \mathbb{N}$ and $\bar{\alpha} \in] -1, 1[^{n+1}$. The restriction to $[-1, 1]^{n+1}$ of this function (which is clearly $\mathbb{R}_{\text{an}(n+1)}$ -definable), will thus not be $\mathbb{R}_{\text{an}(n)}$ -definable as announced in section 5.

As we noted before Proposition 5.3, the lack of surjectivity of each Φ^s will follow from the lack of surjectivity of some mapping $\Phi_{D_s D_{s+1}}^s$ between finite dimensional spaces.

More precisely, if we fix a s and a D , the function

$$E \mapsto \dim(\mathbb{R} \times (\mathbb{R}_{0,E}[X_1, \dots, X_n]^{q(s)})^{p(s)+1})$$

is a polynomial of degree n in E , whereas

$$E \mapsto \dim(\mathbb{R}_{D,E}[Y_1, \dots, Y_{n+1}])$$

is a polynomial of degree $n+1$.

We thus can build an increasing sequence of integers (D_s) such that

$$\dim(\mathbb{R} \times (\mathbb{R}_{0,D_{s+1}}[X_1, \dots, X_n]^{q(s)})^{p(s)+1}) + n + 1$$

is smaller than

$$\dim(\mathbb{R}_{D_s, D_{s+1}}[Y_1, \dots, Y_{n+1}]),$$

for each s .

Suppose we have built for $r < s$ some $P_r \in \mathbb{R}_{D_r, D_{r+1}}[Y_1, \dots, Y_{n+1}]$ and $\eta_r > 0$ such that

$$(A_r): \forall t < r, \|P_r\|_\infty \leq \frac{\eta_t}{2^r (D_{r+1}!)^{n+1}}$$

(B_r): the ball of center P_r and radius $\eta_r S_r$ (where S_r is such that $\forall \bar{\alpha} \in]-1, 1[^{n+1}$, $S_r \geq \|(T_{D_r D_{r+1}}^{n+1} \circ \tau_{\bar{\alpha}})^{-1}\|_\infty$; see remark 6.2) does not meet the image of $\rho_r : (\alpha, \xi) \mapsto (T_{D_r D_{r+1}}^{n+1} \circ \tau_{\bar{\alpha}})^{-1} \circ \Phi_{D_r D_{r+1}}^r(\xi)$ (where α ranges in $] - 1, 1[^{n+1}$ and ξ in \widetilde{M}_s).

We can then chose a $P_s \in \mathbb{R}_{D_s, D_{s+1}}[Y_1, \dots, Y_{n+1}]$ and $\eta_s > 0$ satisfying (A_s) and (B_s):

let $\delta = \min\{\frac{\eta_t}{2^r (D_{r+1}!)^{n+1}}; t < s\}$; by dimensional inequality of source and image space (due to the choice of D_{s+1}) and rationality of $\rho_s : (\alpha, \xi) \mapsto (T_{D_s D_{s+1}}^{n+1} \circ \tau_{\bar{\alpha}})^{-1} \circ \Phi_{D_s D_{s+1}}^s(\xi)$, we know that the image of ρ_s is nowhere dense in $\mathbb{R}_{D_s, D_{s+1}}[Y_1, \dots, Y_{n+1}]$.

We thus can chose a P_s and η_s such that $\|P_s\| < \delta$ and

$$B(P_s; \eta_s S_s) \cap \rho_s(]-1, 1[^{n+1} \times \widetilde{M}_s) = \emptyset.$$

Let $h(Y_1, \dots, Y_{n+1})$ be the formal series $\sum_s P_s(Y_1, \dots, Y_{n+1})$.

We easily get from the inequalities (A_r) that h is the power expansion of an analytic function on \mathbb{R}^{n+1} .

Let $\bar{\alpha}$ be a point in $] - 1, 1[^{n+1}$.

From condition (B_r) we get that

$$(T_{D_r D_{r+1}}^{n+1} \circ \tau_{\bar{\alpha}})(B(T_{D_r D_{r+1}}^{n+1} h; \eta_r S_r)) \cap T_{D_r D_{r+1}}^{n+1} \Phi_r(M_r) = \emptyset$$

and then by definition of S_r ,

$$B((T_{D_r D_{r+1}}^{n+1} \circ \tau_{\bar{\alpha}} \circ T_{D_r D_{r+1}}^{n+1}) h; \eta_r) \cap T_{D_r D_{r+1}}^{n+1} \Phi_r(M_r) = \emptyset.$$

By (A_s) for $s > r$, we get from Proposition 6 that

$$\|(T_{D_r D_{r+1}}^{n+1} \circ \tau_{\bar{\alpha}}) h - (T_{D_r D_{r+1}}^{n+1} \circ \tau_{\bar{\alpha}} \circ T_{D_r D_{r+1}}^{n+1}) h\|_{\infty} \leq \eta_r;$$

thus

$$(T_{D_r D_{r+1}}^{n+1} \circ \tau_{\bar{\alpha}}) h \notin T_{D_r D_{r+1}}^{n+1} \Phi_r(M_r).$$

Hence

$$\tau_{\bar{\alpha}} h \notin \Phi_r(M_r).$$

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